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On the Truncation of Systems with Non-Summable Interactions*

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In this note we consider long-range q-states Potts models on $\mathbf{Z}^d$, $d \geq 2$. For various families of non-summable ferromagnetic pair potentials $\phi(x) \geq 0$, we show that there exists, for all inverse temperature $\beta > 0$, an integer N such that the truncated model, in which all interactions between spins at distance larger than N are suppressed, has at least q distinct infinite-volume Gibbs states. This holds, in particular, for all potentials whose asymptotic behaviour is of the type $\phi(x) \sim \|x\|^{-\alpha}$, $0 \leq \alpha \leq d$. These results are obtained using simple percolation arguments.

KEY WORDS: Long-range Potts model, percolation, phase transition, truncation, non-summable interaction.

1. INTRODUCTION

The existence of phase transitions, in lattice systems of equilibrium statistical mechanics, is a mathematically well posed problem in the canonical framework of infinite-volume Gibbs states with *summable* interactions[(19)]. For such interactions, the Dobrushin Uniqueness Theorem guarantees uniqueness of the Gibbs state at high temperature, and non-uniqueness at low temperature is usually obtained as a consequence of high sensitivity to boundary conditions. The two-dimensional Ising model, for example, can be prepared in distinct thermodynamic states by taking the thermodynamic limit along sequences of boxes with different boundary conditions. The combination of these high and low temperature behaviours leads to a well

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defined finite critical temperature $T_c > 0$. In general, the low temperature phases, $0 < T < T_c$, are described by local deviations from the ground state configurations of the hamiltonian, at which the measure concentrates when $T \to 0$.

For *non-summable* interactions, infinite-volume Gibbs states and the thermodynamic limit are not defined. Physical quantities like the free energy or pressure density don't exist, due to the fact that in the limit of large volumes, the energy of the system grows faster than its size. Even the use of boundary conditions, in finite volumes, poses problems. The pathological behaviour of non-summable systems was well described by Dyson, at a heuristic level in the case of ferromagnetic Ising models[12]: "*When [the potential is non-summable] there is an infinite energy-gap between the ground states and all other states, so that the system is completely ordered at all finite temperatures, and there can be no question of a phase transition*". In other words, since the cost for flipping any given spin is infinite, the temperature, even very high, can never obtain to create local deviations from the ground states, as the ones described above in the case of summable potentials: at any temperature the system is "frozen" in one of its ground states, and no critical temperature can be defined. Therefore one can infer that infinitely large lattice systems with non-summable potentials have, independently of the method used to describe them, trivial thermodynamic behaviour. Nevertheless, various rescalings have been considered in the literature, in view of obtaining non-trivial quantities (called pseudo-densities) in the thermodynamic limit. See for example[23] for a different scaling of the free energy of gravitational and electrostatic particle systems, or[9,10,31], where mean field versions of ferromagnetic spin models with non-summable interactions have been studied numerically.

In the present note, we study non-summable systems by using a simpler approach which is the following. Let $\mu_{\phi,\Lambda}$ denote the Gibbs distribution of the system with ferromagnetic pair interaction ϕ in finite volume Λ. Since nothing can be said, in the sense of weak convergence, about the existence of the thermodynamic limit

$$\lim_{\Lambda} \mu_{\phi,\Lambda},$$

we first *truncate* the potential ϕ by suppressing interactions between points at distance larger than N:

$$\phi_N(x) := \begin{cases} \phi(x) & \text{if } \|x\| \leq N, \\ 0 & \text{if } \|x\| > N, \end{cases} \tag{1}$$

and then study the double limiting procedure

$$\lim_{N\to\infty} \lim_{\Lambda} \mu_{\phi_N,\Lambda}.$$

Since ϕ_N has finite range, the infinite-volume truncated model with measure $\mu_{\phi_N} = \lim_\Lambda \mu_{\phi_N,\Lambda}$ is well defined. Observe that in $d = 1$, μ_{ϕ_N} does not show interesting dependence on boundary conditions, since one dimensional finite-range models don't exhibit phase transitions. Nevertheless, for $d \geq 2$ and when ϕ is non-summable, the phenomenon which we observe is the following: for *any temperature, the measure* μ_{ϕ_N} *becomes sensitive to boundary conditions once* N *is large enough (but finite)*. This means that if ϕ has a finite set of ground state configurations indexed by s, and if we denote by $\mu^s_{\phi_N} = \lim_\Lambda \mu^s_{\phi_N,\Lambda}$ the Gibbs state obtained by taking the thermodynamic limit along a sequence of boxes with boundary condition s, then $\mu^s_{\phi_N} \neq \mu^{s'}_{\phi_N}$ when N is sufficiently large. Moreover, in the limit $N \to \infty$, each $\mu^s_{\phi_N}$ converges weakly to δ_s, the Dirac measure concentrated on the ground state configuration s. This concentration phenomenon in the limit $N \to \infty$ *at any fixed temperature* $T > 0$ is thus similar to the one discussed above for summable interactions in the limit $T \to 0$ for systems in dimensions $d \geq 2$, and is in agreement with Dyson's heuristic description of non-summable interactions. Notice that in our approach, the interaction between any pair of spins is restored in the limit $N \to \infty$, and no mean field rescaling is ever used.

In Section 2 we show that the scenario presented above indeed occurs for the ferromagnetic Potts model on $\mathbf{Z}^d$, $d \geq 2$, for various families of non-summable potentials. Our results include, for instance, all potentials with slow algebraic decay (see Theorem 1):

$$\phi(x) \sim \frac{1}{\|x\|^\alpha} \quad \text{for some} \quad 0 \leq \alpha < d, \tag{2}$$

which are the usual non-summable potentials considered in physics. Nevertheless, our aim is to treat general interactions, which need not be asymptotically regular as in (2), but can have an irregular structure. We also give two results for *sparse* interactions, which are of independent interest. In Section 3 we give the inequality which allows to study this problem via independent long-range percolation and then reformulate and prove all our results in this setting. As will be seen, the proofs are simple geometric arguments. In Section 4 we conclude with some general remarks.

This work originated with the study of the problem of truncation in independent long-range percolation[16] (see refs.[3,28,30] for other cases treated in the literature). Therefore, the results presented in Section 3, independently of the rest of the paper, give more particular cases where this problem can be solved. Nevertheless, our techniques don't rely on multiscale analysis or renormalisation such as those of refs.[3,28,30], and allow irregular asymptotic behaviour of the edge probabilities.

2. LONG-RANGE POTTS FERROMAGNET

We consider the lattice $\mathbf{Z}^d$, $d \geq 2$, with the norm $\|x\| = \max_{k=1,\ldots,d} |x_k|$. Interactions are defined via a ferromagnetic potential, which is any function

$\phi : \mathbf{Z}^d \setminus \{0\} \to [0, +\infty)$ such that $\sup_{x \neq 0} \phi(x) < +\infty$, with the symmetry

$$\phi(x) = \phi(y) \quad \text{when} \quad \|x\| = \|y\|. \tag{3}$$

Let $N \in \mathbf{N}$. To each potential ϕ can be associated a truncated potential ϕ_N, defined as in (1). In the q-state Potts model, $q \geq 2$ is any fixed integer, and at each site $x \in \mathbf{Z}^d$ lives a spin $\sigma_x \in \{1, 2, \ldots, q\}$. When $q = 2$ it thus reduces to the Ising model. Spin configurations are elements of $\Omega = \{1, 2, \ldots, q\}^{\mathbf{Z}^d}$. Consider a finite box $\Lambda_L = [-L, +L]^d \cap \mathbf{Z}^d$, $L \geq 1$. For $\sigma \in \Omega_{\Lambda_L} = \{1, 2, \ldots, q\}^{\Lambda_L}$, the truncated Potts Hamiltonian with boundary condition $\eta \in \Omega$ is given by

$$H^{\eta}_{N,\Lambda_L}(\sigma) = -\sum_{\substack{\{x,y\} \subset \Lambda_L \\ x \neq y}} \phi_N(x-y)\delta(\sigma_x, \sigma_y) - \sum_{x \in \Lambda_L, y \in \Lambda_L^c} \phi_N(x-y)\delta(\sigma_x, \eta_y),$$

where $\delta(a, b) = 1$ if $a = b$, 0 otherwise. We will mainly be interested in considering the pure s boundary condition, in which $\eta_j = s$ for all $j \in \mathbf{Z}^d$. We have, with some abuse of notation,

$$H^s_{N,\Lambda_L}(s) = \min_{\sigma \in \Omega_{\Lambda_L}} H^s_{N,\Lambda_L}(\sigma).$$

Therefore, we also call the pure configurations s ground state configurations. On Ω_{Λ_L}, the truncated Gibbs measure at inverse temperature $\beta > 0$ with pure s boundary condition is defined by:

$$\mu^{\beta,s}_{\phi_N,\Lambda_L}(\sigma) := \frac{1}{Z^{\beta,s}_{\phi_N,\Lambda_L}} \exp\big(-\beta H^s_{N,\Lambda_L}(\sigma)\big),$$

where $Z^{\beta,s}_{\phi_N,\Lambda_L}$ is a normalizing factor. Let $\mathcal{F}$ be the σ-algebra on Ω generated by cylinder events. We consider the infinite-volume Gibbs measures $\mu^{\beta,s}_{\phi_N}$ on $(\Omega, \mathcal{F})$, obtained by taking limits along an increasing sequence of boxes[1] (this limit is to be understood in the sense of subsequences):

$$\mu^{\beta,s}_{\phi_N}(A) := \lim_{L \to \infty} \mu^{\beta,s}_{\phi_N,\Lambda_L}(A) \quad \forall A \in \mathcal{F}.$$

A phase transition occurs in the truncated model if $\mu^{\beta,s}_{\phi_N} \neq \mu^{\beta,s'}_{\phi_N}$ for $s' \neq s$.
Let $B_N := \Lambda_N \setminus \{0\}$. When ϕ is summable, i.e. when

$$\sum_{x \neq 0} \phi(x) := \lim_{N \to \infty} \sum_{x \in B_N} \phi(x) \tag{4}$$

[1] Here we extend $\mu^{\beta,s}_{\phi_N,\Lambda_L}$ to a measure on $(\Omega, \mathcal{F})$ in the standard way:

$$\mu^{\beta,s}_{\phi_N,\Lambda_L}(A) := \sum_{\sigma \in \Omega_{\Lambda_L}} \mu^{\beta,s}_{\phi_N,\Lambda_L}(\sigma) 1_A(\sigma \cdot s) \quad \forall A \in \mathcal{F},$$

where the configuration $\sigma \cdot s \in \Omega$ coincides with σ on Λ_L and with s on Λ_L^c.

exists, the untruncated Gibbs measures (with $N = \infty$) $\mu_{\phi}^{\beta,s}$ are well defined, and the problem of knowing whether $\mu_{\phi}^{\beta,s} \neq \mu_{\phi}^{\beta,s'}$ for some $s' \neq s$ depends strongly on the temperature. When ϕ is not summable these measures are not defined, and we study $\mu_{\phi_N}^{\beta,s}$ at large N. We remind that for fixed N, in the limit of very low temperature, $\beta \to \infty$, the typical configurations of $\mu_{\phi_N}^{\beta,s}$ concentrate on the ground state configuration s.[29] When the temperature is fixed and N becomes large, we observe essentially the same phenomenon. In view of the argument of Dyson cited in the Introduction, it is reasonable to believe that at any fixed $\beta > 0$, each of the measures $\mu_{\phi_N}^{\beta,s}$ concentrates, when $N \to \infty$, on a single configuration, which is the ground state s. This is the statement of the following conjecture. Let δ_s denote the Dirac mass on $(\Omega, \mathcal{F})$ concentrated on the ground state configuration s, and write $\mu_{\phi_N}^{\beta,s} \Rightarrow \delta_s$ when $\mu_{\phi_N}^{\beta,s}$ converges weakly to δ_s in the limit $N \to \infty$.

Conjecture 1. ($d \geq 2$) *If $\phi \geq 0$ satisfies (3) and is non-summable, i.e.*

$$\sum_{x \neq 0} \phi(x) = +\infty, \tag{5}$$

then $\mu_{\phi_N}^{\beta,s} \Rightarrow \delta_s$ for all $\beta > 0$ and for all $s \in \{1, 2, \ldots, q\}$.

Observe that $\mu_{\phi_N}^{\beta,s} \Rightarrow \delta_s$ implies $\mu_{\phi_N}^{\beta,s}(\sigma_0 = s) \to 1$ in the limit $N \to \infty$, i.e. a phase transition occurs in the truncated model for large enough N. Since the system is ferromagnetic, the sequence $\left(\mu_{\phi_N}^{\beta,s}(\sigma_0 = s)\right)_{N \geq 1}$ is non-decreasing, but the fact that it converges to 1 is not trivial. A weaker form of the conjecture is:

$$\beta_c(\phi_N) \to 0 \quad \text{when } N \to \infty,$$

where $\beta_c(\phi_N)$ is the critical inverse temperature of the model with potential ϕ_N, i.e.

$$\beta_c(\phi_N) := \inf\left\{\beta > 0 : \mu_{\phi_N}^{\beta,s} \neq \mu_{\phi_N}^{\beta,s'} \text{ for } s \neq s'\right\}.$$

The conjecture is difficult to prove in such generality, since we don't assume any kind of regularity on ϕ. For example, the potential

$$\phi(x) = \begin{cases} \epsilon > 0 & \text{if } \|x\| = k! \text{ for some } k \in \mathbf{N}, \\ 0 & \text{otherwise,} \end{cases}$$

which will enter in the family of interactions considered in Theorem 2, satisfies the hypothesis of the conjecture. Usual perturbation techniques, such as Pirogov-Sinai Theory[29], are of no use for studying the truncated version of this kind of potential, since the domain of validity for the temperature shrinks to zero when the range of interaction, here N, grows.

Remark 1. Observe that since the Gibbs state of any one-dimensional model with finite-range interactions is always unique (see for example Theorem 8.39 in[19]), Conjecture 1 is *false* in dimension 1.

For $d \geq 2$, our first result shows that the conjecture is valid under some assumption on the speed of divergence of the series (5).

Theorem 1. ($d \geq 2$) *If $\phi \geq 0$ satisfies (3), and if (5) diverges faster than logarithmically, i.e.*

$$\limsup_{N\to\infty} \frac{1}{\log|B_N|} \sum_{x\in B_N} \phi(x) = +\infty, \tag{6}$$

then $\mu^{\beta,s}_{\phi_N} \Rightarrow \delta_s$ for all $\beta > 0$ and for all $s \in \{1, 2, \ldots, q\}$.

Remark 2. As can be seen easily, condition (6) is satisfied by all potentials which have slow algebraic decay, as in (2):

$$\liminf_{\|x\|\to\infty} \|x\|^{\alpha}\phi(x) > 0 \quad \text{for some} \quad 0 \leq \alpha < d.$$

We shall see later, in Remark 3, that for such potentials the range $0 \leq \alpha < d$ can be extended to $0 \leq \alpha \leq d$, using the multiscale analysis of[3].

Sparse Interactions. We also give two results for potentials which don't have the symmetry (3). Namely, we consider interactions only along directions parallel to the coordinate axis e_i, $i = 1, \ldots, d$, where $e_1 = (1, 0, \ldots, 0)$, $e_2 = (0, 1, 0, \ldots, 0), \ldots, e_d = (0, 0, \ldots, 1)$. That is, we are given a sequence $(\phi_n)_{n\geq 1}$, $\phi_n \geq 0$, and[2]

$$\phi(x) = \begin{cases} \phi_{\|x\|} & \text{if } x \text{ is parallel to some } e_i,\, i = 1, 2, \ldots, d, \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

In $d = 2$, for example, there are only vertical and horizontal couplings. For potentials of the form (7), assumption (5) of Conjecture 1 becomes:

$$\sum_{n\geq 1} \phi_n = +\infty. \tag{8}$$

The first result is for sequences $(\phi_n)_{n\geq 1}$ which don't converge to zero:

[2] The reader should pay attention to the following: we use n to index elements of the sequence $(\phi_n)_{n\geq 1}$, whereas N is used as the parameter of truncation for ϕ_N.

Theorem 2. $(d \geq 2)$. *If*

$$\limsup_{n\to\infty} \phi_n > 0, \tag{9}$$

then $\mu_{\phi_N}^{\beta,s} \Rightarrow \delta_s$ *for all* $\beta > 0$ *and for all* $s \in \{1, 2, \ldots, q\}$.

Notice that (9) implies (8), but with no information on the speed of divergence. Our second result is where we prove our conjecture under the general condition (8), with no assumption on the speed of divergence, but only in dimensions three or more:

Theorem 3. $(d \geq 3)$. If

$$\sum_{n\geq 1} \phi_n = +\infty, \tag{10}$$

then $\mu_{\phi_N}^{\beta,s} \Rightarrow \delta_s$ for all $\beta > 0$ and for all $s \in \{1, 2, \ldots, q\}$.

The following proposition gives a criterion that will be used for showing weak convergence towards the Dirac masses δ_s. The proof is postponed to Section 5.

Proposition 1. *For all* $\beta > 0$ *and all* $s \in \{1, 2, \ldots, q\}$, *the following holds:* $\mu_{\phi_N}^{\beta,s} \Rightarrow \delta_s$ *when* $N \to \infty$ *if and only if*

$$\lim_{N\to\infty} \mu_{\phi_N}^{\beta,s}(\sigma_0 = s) = 1. \tag{11}$$

3. INDEPENDENT LONG-RANGE PERCOLATION

To show that the truncated q-states Potts model at inverse temperature $\beta > 0$ exhibits a phase transition, it is actually sufficient to show that

$$\mu_{\phi_N}^{\beta,s}(\sigma_0 = s) > \frac{1}{q}. \tag{12}$$

To obtain (11), which is stronger than (12), we shall reformulate our problem in the framework of long-range independent percolation.

Consider the graph $(\mathbf{Z}^d, \mathcal{E}^d)$, $d \geq 1$, where $\mathcal{E}^d$ is the set of all unoriented edges $e = \{x, y\} \subset \mathbf{Z}^d \times \mathbf{Z}^d, x \neq y$. Edge configurations are elements $\omega \in \{0, 1\}^{\mathcal{E}^d}$. For a given function $p : \mathbf{Z}^d \backslash \{0\} \to [0, 1]$ with

$$p(x) = p(y) \quad \text{when} \quad \|x\| = \|y\|, \tag{13}$$

called edge probability, we consider the independent long-range percolation process in which each edge $e = \{x, y\}$ is open ($\omega(e) = 1$) with probability $p(x - y)$, and closed ($\omega(e) = 0$) with probability $1 - p(x - y)$, independently of other edges. This process is described by the product measure on the σ-field on $\{0, 1\}^{\mathcal{E}^d}$ generated by cylinders, given by

$$P = \prod_{e \in \mathcal{E}^d} \mu_e, \tag{14}$$

where $\mu_e(\omega(e) = 1) = p(x - y)$ is a Bernoulli measure on $\{0, 1\}$, independent of the state of other edges. Observe that P is well-defined even when p is non-summable. Define the truncated edge probability

$$p_N(x) := \begin{cases} p(x) & \text{if } \|x\| \leq N, \\ 0 & \text{if } \|x\| > N, \end{cases}$$

and denote by P_N the truncated product measure defined as in (14) with p_N instead of p. We shall be interested in the percolation probability $P_N(0 \leftrightarrow \infty)$, which is the probability of the event in which there exists, in the truncated model, a path of open edges connecting the origin to infinity. When $P_N(0 \leftrightarrow \infty) > 0$, we say the truncated system percolates.

As can be seen in the following proposition, the percolation probability is a relevant quantity for showing all our results for the Potts model, once the edge probability is well chosen in function of the potential ϕ.

Proposition 2. (Fortuin,[13,14]). *Define $p(x)$ by*

$$p(x) := \frac{1 - e^{-2\beta\phi(x)}}{1 + (q - 1)e^{-2\beta\phi(x)}}. \tag{15}$$

Then the magnetisation of the truncated long-range q-states Potts model with potential $\phi_N(x)$ at temperature β and the probability of percolation of the origin in the independent truncated long-range percolation process with edge probabilities $p_N(x)$ are related by the following inequality:

$$\mu_{\phi_N}^{\beta,s}(\sigma_0 = s) \geq \frac{1}{q} + \frac{q - 1}{q} P_N(0 \leftrightarrow \infty). \tag{16}$$

Using (12, 16) shows that percolation in the truncated independent model implies phase transition in the truncated Potts model. This holds once the potential ϕ and the probability p are related by (15), which is well suited for our purposes since (15) implies that ϕ and p are bound to have the same asymptotic behaviour: as can be seen, there exist two positive functions $C_\pm = C_\pm(\beta, q)$ such that

$$C_- \phi(x) \leq p(x) \leq C_+ \phi(x) \quad \forall x \neq 0.$$

In particular, non-summability of ϕ implies non-summability of p. Although it does not appear exactly in this form in the literature, (16) is standard, and can be obtained via the random cluster representation of the measure $\mu^{\beta,s}_{\phi_N}$, and its domination properties with respect to Bernoulli product measures. We refer to the literature[15,13,14,1,20] for details.

By Proposition 2, long-range independent percolation can be used to show all the results stated previously for the Potts model. We therefore reformulate and prove the equivalent of the results of Section 2 in the context of independent percolation: using Proposition 2, Theorem 1 (resp. 2, 3)will follow from Theorem 4 (resp. 5, 6).

For percolation, our conjecture is the following: when 13 holds and

$$\sum_{x\neq 0} p(x) = +\infty, \tag{17}$$

then $\lim_{N\to\infty} P_N(0 \leftrightarrow \infty) = 1$. In one dimension, percolation with edge probabilities satisfying 17 was studied in[21], where it was shown in particular that $P(0 \leftrightarrow \infty) = 1$. Our first result for percolation is

Theorem 4. ($d \geq 2$) *There exists* $c = c(d) > 0$ *such that if (13) holds and if*

$$\limsup_{N\to\infty} \frac{1}{\log |B_N|} \sum_{x\in B_N} p(x) \geq c, \tag{18}$$

then $\lim_N P_N(0 \leftrightarrow \infty) = 1$.

Remark 3. *As can be seen easily, 18 is satisfied when*

$$\lambda := \liminf_{\|x\|\to\infty} \|x\|^{\alpha} p(x) > 0, \quad \text{for some} \quad 0 \leq \alpha < d. \tag{19}$$

In fact Theorem 4 can be obtained, under condition 19, using the multiscale analysis of Berger,[3] *for all* $0 \leq \alpha \leq d$. *Our condition 18 allows the function* $p(\cdot)$ *to behave in some irregular manner, but does not cover the case* $\alpha = d$, *unless* λ *is assumed sufficiently large.*

Proof of Theorem 4: The proof is a simple (almost trivial) blocking argument. Fix N large. For each $s \in S := \{(s_1, \dots, s_d) : s_i = \pm 1\}$, define the quadrant $Q^N_s(0) := \{y \in \mathbf{Z}^d : 0 < s_i y_i \leq N, \ i = 1, 2, \dots, d\}$. For any $x \in \mathbf{Z}^d$, let $Q^N_s(x) := x + Q^N_s(0)$. A site x is good if there exists, for all $s \in S$, a site

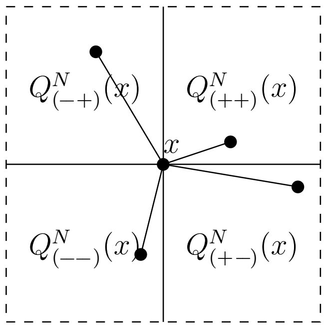

Fig. 1. Illustration of a good site x in the two-dimensional case.

$y \in Q_s^N(x)$ such that the edge $\{x, y\}$ is open (see Figure 1). We have

$$
\begin{aligned}
P_N(x \,\text{is good}) &= \prod_{s \in S} \Big[1 - \prod_{y \in Q_s^N(x)} (1 - p(x-y)) \Big] \\
&= \Big[1 - \prod_{y \in Q_s^N(0)} (1 - p(y)) \Big]^{|S|} \quad \forall s \in S \\
&\geq \Big[1 - \exp\Big(- \sum_{y \in Q_s^N(0)} p(y) \Big) \Big]^{|S|} \quad \forall s \in S \\
&\geq \Big[1 - \exp\Big(- c_1 \sum_{y \in B_N} p(y) \Big) \Big]^{|S|}, \qquad (20)
\end{aligned}
$$

where we used the inequality $\log(1-t) \leq -t$ valid for all $t < 1$, the symmetry (13), and $c_1 > 0$ is a constant that depends only on the dimension. Next, consider a partition of $\mathbf{Z}^d$ into disjoint blocks of linear size $3N$, obtained by translates of the block $C^N(0) := [0, 3N)^d \cap \mathbf{Z}^d$. That is, each block of this partition is of the form $C^N(z) = 3zN + C^N(0)$, for some renormalized vertex $z \in \mathbf{Z}^d$. We say a block $C^N(z)$ is good if each $x \in C^N(z)$ is good. Now for two points $x \neq x'$, the events $\{x \text{ is good}\}$ and $\{x' \text{ is good}\}$ are not necessarily independent, but since they are increasing, FKG inequality gives

$$
P_N(C^N(z) \text{ is good}) \geq \prod_{x \in C^N(z)} P_N(x \text{ is good}).
$$

For small enough $\epsilon > 0$ we have $1 - \epsilon \geq e^{-2\epsilon}$. Using $|C^N(z)| \leq c_2|B_N|$ and (20) we thus get, when N is large enough,

$$P_N(C^N(z) \text{ is good}) \geq \exp\Big[- 2c_2|S||B_N| \exp\Big(- c_1 \sum_{y\in B_N} p(y)\Big)\Big]$$
$$= \exp\Big[- 2c_2|S| \exp\Big(\log|B_N| - c_1 \sum_{y\in B_N} p(y)\Big)\Big].$$

If (18) holds with a well chosen constant $c > 0$ we get

$$\limsup_{N\to\infty} P_N(C^N(z) \text{ is good}) = 1,$$

i.e. there exists a sequence $\delta_k \searrow 0$ and a diverging sequence $N_1, N_2, \ldots$ such that

$$P_{N_k}(C^{N_k}(z) \text{ is good}) \geq 1 - \delta_k \quad \forall k. \tag{21}$$

Notice that for each k the process $(X^k_z)_{z\in\mathbf{Z}^d}$ defined by

$$X^k_z := \begin{cases} 1 & \text{if } C^{N_k}(z) \text{ is good,} \\ 0 & \text{otherwise,} \end{cases} \tag{22}$$

is 1-dependent, i.e. X^k_z and $X^k_{z'}$ are independent when $\|z - z'\| > 1$. By a Theorem of Liggett, Stacey and Schonmann,[27] $(X^k_z)_{z\in\mathbf{Z}^d}$ stochastically dominates an independent Bernoulli process $(Z^k_z)_{z\in\mathbf{Z}^d}$ of parameter $\rho_k > 0$, and (21) implies $\lim_{k\to\infty} \rho_k = 1$. Take k large enough such that $\rho_k > p_c(\mathbf{Z}^d, \text{site})$, where $p_c(\mathbf{Z}^d, \text{site})$ is the critical threshold of Bernoulli nearest-neighbour site percolation. For such k, the process $(Z^k_z)_{z\in\mathbf{Z}^d}$ is supercritical and by domination there exists an infinite cluster of good boxes. It is easy to see that any infinite connected component of good boxes yields an infinite connected component of sites of the original lattice. By taking N large one can thus make $P_N(0 \leftrightarrow \infty)$ arbitrarily close to 1. $\square$

Remark 4. Criterion (18) concerns the behaviour of the sum $\sum_{x\in B_N} p(x)$ for large N, *and not the details of the function* $p(\cdot)$. This has an interesting consequence, as the following discussion shows. Assume p is such that (18) holds. Then Theorem 4 guarantees the existence of some N such that the system whose edges $e = \{x, y\}$ are all of size at most $\|x - y\| \leq N$, with edge probabilities $p(\cdot)$, satisfies $P_N(0 \leftrightarrow \infty) > 0$. As seen in the proof, the integer N is fixed once the sum of the edge probabilities passes a given value K_N, i.e.

$$\sum_{x\in B_N} p(x) \geq K_N.$$

Now, observe that the function p can be modified inside B_N, but as long as the *sum* is preserved, the percolation probability remains positive. For example, if

$\pi : B_N \to B_N$ is any permutation preserving the symmetry $\|\pi(x)\| = \|\pi(y)\|$ for $\|x\| = \|y\|$, then

$$\sum_{x \in B_N} p(\pi(x)) = \sum_{x \in B_N} p(x) \geq K_N,$$

and so the truncated system with edge probability $p(\pi(\cdot))$ also percolates. This comment suggests that the sum $\sum_{x \in B_N} p(x)$, rather than the individual edge probabilities, is the relevant parameter in the study of percolation in the truncated model. This property can also be easily verified for long-range percolation on trees.

Sparse Connections. We now give two results concerning systems where connections are not isotropic, and we will consider the case where a connection can be opened between two sites x, y only if these lie on a same coordinate axis. That is, $p(x) \neq 0$ only if x is parallel to one of the coordinate axes e_i, $i = 1, \ldots, d$. In this case the probabilities $p(x)$ are determined by a sequence $(p_n)_{n \geq 1}$, $p_n \in [0, 1]$, and

$$p(x) = \begin{cases} p_{\|x\|} & \text{if } x \text{ is parallel to some } e_i, i = 1, 2, \ldots, d, \\ 0 & \text{otherwise.} \end{cases} \tag{23}$$

We expect that

$$\sum_{n \geq 1} p_n = +\infty \tag{24}$$

implies $\lim_N P_N(0 \leftrightarrow \infty) = 1$. A particular case of (24) in which nothing is assumed about the speed of divergence of the series is the following.

Theorem 5. $(d \geq 2)$ *If*

$$\limsup_{n \to \infty} p_n > 0 \tag{25}$$

then $\lim_N P_N(0 \leftrightarrow \infty) = 1$.

Since this result has already appeared in ref.[16] we shall only remind the reader of the strategy of the proof, which is very different, in spirit, from that of Theorem 4. We consider the two-dimensional case $d = 2$. Nevertheless, the core of the proof is to use properties of nearest-neighbour percolation in high dimensions d_*. Denote by $p_c(\mathbf{Z}^{d_*})$ the percolation threshold of nearest-neighbour Bernoulli edge percolation on $\mathbf{Z}^{d_*}$. It was shown by Kesten[25] that

$$p_c(\mathbf{Z}^{d_*}) \to 0 \quad \text{when} \quad d_* \to \infty. \tag{26}$$

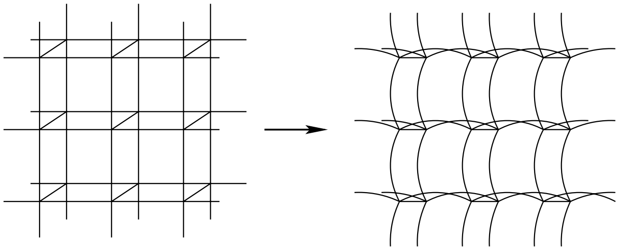

Fig. 2. The embedding of the slab $\{1, 2\} \times \mathbf{Z}^2 \subset \mathbf{Z}^3$ in $(\mathbf{Z}^2, \mathcal{E}^2)$.

Then, let $\{1, 2, \ldots, L\}^{d_*-2} \times \mathbf{Z}^2$ denote the slab of thickness L in $\mathbf{Z}^{d_*}$. It was shown by Grimmett and Marstrand [22] that the slab percolation threshold satisfies

$$p_c(\{1, 2, \ldots, L\}^{d_*-2} \times \mathbf{Z}^2) \to p_c(\mathbf{Z}^{d_*}) \quad \text{when} \quad L \to \infty. \tag{27}$$

Now call 2ϵ the lim sup in (25), and consider some diverging sequence $n_1, n_2, \ldots$ for which $p_{n_k} \geq \epsilon$. By (26) and (27) there exists a dimension d_* and an integer L (both depending on ϵ) such that for all k,

$$p_{n_k} \geq \epsilon > p_c(\{1, 2, \ldots, L\}^{d_*-2} \times \mathbf{Z}^2). \tag{28}$$

It is then clear how to pursue: we *embed* the slab $\{1, 2, \ldots, L\}^{d_*-2} \times \mathbf{Z}^2$ in $(\mathbf{Z}^2, \mathcal{E}^2)$, using edges of sizes taken in the set $\{n_1, n_2, \ldots\}$. Here, N must be taken large enough. The point is that we only need a *finite* number of sizes, and that by (28) each edge of the embedded graph has probability at least $\epsilon > p_c(\{1, 2, \ldots, L\}^{d_*-2} \times \mathbf{Z}^2)$ of being open. This guarantees that the independent truncated process on this graph is supercritical, hence contains with probability one an infinite cluster: $P_N(0 \leftrightarrow \infty) > 0$. The simplest case for which the embedding can be easily understood is when $d_* = 3$ and $L = 2$, which we illustrated on Figure 2. For higher dimension, for example in the case where $d_* = 5$, we have represented on Figure 3 an embedding of the cube $\{1, 2\}^3$ in $(\mathbf{Z}^2, \mathcal{E}^2)$, which shows what must be done in the general case. The formal embedding of the slab $\{1, 2, \ldots, L\}^{d_*-2} \times \mathbf{Z}^2$ can be found in[16]. We leave it as an exercise to the reader to show that (26) can be used again to show that $\lim_N P_N(0 \leftrightarrow \infty) = 1$.

Remark 5. In order to deduce Theorem 2 from Theorem 5, one must estimate

$$\limsup_{n\to\infty} p_n = \limsup_{n\to\infty} \frac{1 - e^{-2\beta\phi_n}}{1 + (q-1)e^{-2\beta\phi_n}} \geq C_-(\beta, q) \limsup_{n\to\infty} \phi_n.$$

Observe that $C_-(\beta, q)$ goes to zero when $\beta \searrow 0$. Therefore, the dimension d_* and the thickness L of the slab used in the proof above diverges for large temperatures.

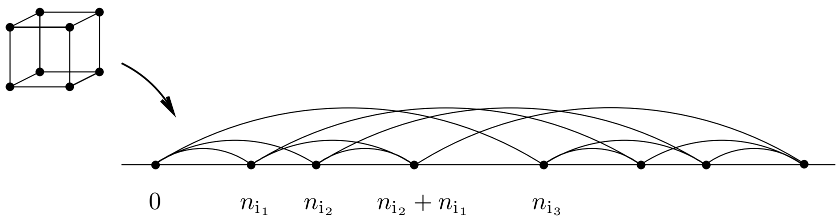

Fig. 3. The embedding of the cube $\{1, 2\}^3$ on the positive axis of $(\mathbf{Z}^2, \mathcal{E}^2)$, using long-range edges. We have chosen $n_{i_1} := n_1$, $n_{i_2} := \min\{n_k : n_k > n_{i_1}\}$, and $n_{i_3} := \min\{n_k : n_k > n_{i_1} + n_{i_2}\}$, in order to avoid overlapping. Then, an infinite number of copies of this cube must be glued together, to form the slab $\{1, 2\}^3 \times \mathbf{Z}^2$ embeded in $(\mathbf{Z}^2, \mathcal{E}^2)$.

Remark 6. Theorem 2 follows from Theorem 5 and Proposition 2. Recently, Bodineau(7) has proved the analogue of the result of Grimmett and Marstrand(22) which we used in ref.(27) for the percolation threshold of the random cluster measure associated to the Ising model ($q = 2$). Therefore, using the main result of ref.(7) and the same embedding as above, one can obtain a more direct proof of Theorem 2 for the case $q = 2$, without using Proposition 2.

Theorem 6. *($d \geq 3$) If*

$$\sum_{n\geq 1} p_n = +\infty \tag{29}$$

then $\lim_N P_N(0 \leftrightarrow \infty) = 1$.

Proof: The proof relies on the fact that sequences p_n that satisfy (29) already imply percolation in one dimension. So for a while we consider long-range percolation on $(\mathbf{Z}, \mathcal{E}^1)$, and denote by P^1 the product measure on $\{0, 1\}^{\mathcal{E}^1}$ associated to the sequence $(p_n)_{n\geq 1}$. Let $\theta := P^1(0 \leftrightarrow \infty)$. By ref.(21), $\theta = 1$. For simplicity we first assume that there exists, for all $m \in \mathbf{Z}$, a sequence $n_1 = 0, n_2, \ldots, n_k = m$ such that $p_{|n_{i+1}-n_i|} > 0$. This implies, by ref.(2), that the infinite cluster is unique.

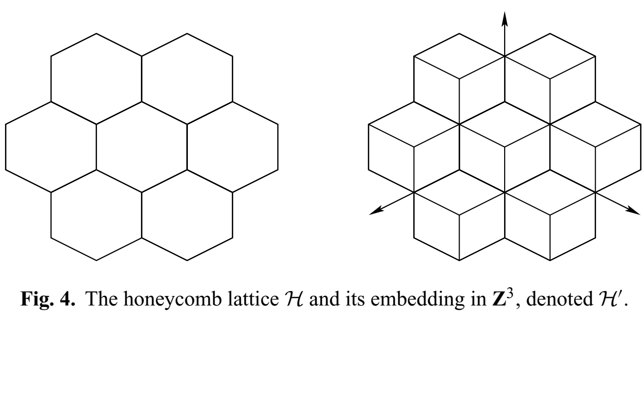

Fig. 4. The honeycomb lattice $\mathcal{H}$ and its embedding in $\mathbf{Z}^3$, denoted $\mathcal{H}'$.

We can thus write

$$\begin{aligned} P^1(0 \leftrightarrow 1) &\geq P^1(0 \leftrightarrow \infty, 1 \leftrightarrow \infty) \\ &\geq P^1(0 \leftrightarrow \infty) P^1(1 \leftrightarrow \infty) = \theta^2, \end{aligned} \tag{30}$$

where we have used, in this order, uniqueness of the infinite cluster, FKG inequality, and translation invariance. For $a < b$ let $T_L(a, b) := [a - L, b + L]$. Then $\{0 \leftrightarrow 1 \text{ in } T_L(0, 1)\} \nearrow \{0 \leftrightarrow 1\}$ when $L \to \infty$. Therefore for all $\delta > 0$ there exists L_δ such that for $L \geq L_\delta$,

$$P^1(0 \leftrightarrow 1 \text{ in } T_L(0, 1)) \geq \theta^2 - \delta. \tag{31}$$

Since

$$\theta^2 = 1 > p_c(\mathcal{H}), \tag{32}$$

where $p_c(\mathcal{H})$ is the percolation threshold of the honeycomb lattice $\mathcal{H}$, we can take δ small enough such that $\theta^2 - \delta > p_c(\mathcal{H})$, and fix $L \geq L_\delta$.
Back to $d = 3$, consider the embedding of $\mathcal{H}$ in $\mathbf{Z}^3$, denoted $\mathcal{H}'$, depicted on Figure 4.

To each edge $e = \{x, y\} \in \mathcal{H}'$ corresponds a one dimensional subgraph $(\mathbf{Z}(e), \mathcal{E}^1(e)) \subset (\mathbf{Z}^3, \mathcal{E}^3)$ which consists of all the points (and long-range edges) contained in the line supported by e; $(\mathbf{Z}(e), \mathcal{E}^1(e))$ is nothing but a copy of $(\mathbf{Z}, \mathcal{E}^1)$, embedded in $(\mathbf{Z}^3, \mathcal{E}^3)$, containing e. All the previous considerations on $(\mathbf{Z}, \mathcal{E}^1)$ (for example the intervals $T_L(x, y)$) can be adapted in each subgraph $(\mathbf{Z}(e), \mathcal{E}^1(e))$. An important property of the embedding we have chosen is that *the graphs* $(\mathbf{Z}(e), \mathcal{E}^1(e))$, $(\mathbf{Z}(e'), \mathcal{E}^1(e'))$ *associated to two different edges* $e, e' \in \mathcal{H}'$ *have disjoint sets of edges*.

We now define an edge $e = \{x, y\} \in \mathcal{H}'$ to be good if and only if there exists, in $(\mathbf{Z}(e), \mathcal{E}^1(e))$, a path connecting x to y in $T_L(x, y)$. Clearly, edges are good independently and for $N = 2L + 2$,

$$P_N(e \text{ is good}) = P_N^1(x \leftrightarrow y \text{ in } T_L(x, y)) \geq \theta^2 - \delta > p_c(\mathcal{H}).$$

Therefore, there exist infinite paths of good edges, yielding the existence of an infinite cluster on $(\mathbf{Z}^3, \mathcal{E}^3)$ with edges of sizes smaller than N. Clearly, $\lim_N P_N(0 \leftrightarrow \infty) = 1$, which finishes the proof. When the assumption made at the beginning is not satisfied, it suffices to replace $P^1(0 \leftrightarrow 1)$, in (30), by $P^1(0 \leftrightarrow K)$ for a well chosen K. The rest of the proof can be adapted in a straightforward way.
□

Observe that the only place where we used the divergence of the series (29) was to obtain $P^1(0 \leftrightarrow \infty) = 1$. A variant of Theorem 6 can therefore be reformulated under a more abstract condition on the sequence $(p_n)_{n \geq 1}$, which can hold also when the series $\sum_n p_n$ converges:

Theorem 7. $(d \geq 3)$. *Assume the sequence* $(p_n)_{n\geq 1}$ *is such that* $\theta := P^1(0 \leftrightarrow \infty)$ *satisfies* $\theta^2 > p_c(\mathcal{H})$ *and that the one-dimensional infinite cluster is unique. Then* $P_N(0 \leftrightarrow \infty) > 0$ *when* N *is large enough.*

4. FINAL REMARKS

We have considered the problem of truncation in the long-range Potts model with non-summable ferromagnetic interactions, via simple percolation techniques. We have shown that for various families of potentials, a phase transition occurs in the truncated model as soon as the parameter of truncation N is taken sufficiently large. Notice that by Proposition 2 all the existing results on truncation in long-range percolation[(3,28,30)] have their counterpart in the long-range Potts model. We hope that *embeddings*, as those we used in the proofs of Theorems 5 and 6, might be used for possible generalisations since they don't require any particular regularity of the potential/edge probability at infinity, and give some insight into new mechanisms of phase transitions in systems with long-range interactions.

Before ending, we make two remarks concerning the problem of truncation.

4.1. A Mean Field Limit.

As our results show, *infinite* systems with non-summable interactions have trivial dependence on the temperature. In the physics literature, some methods have been used in view of understanding the properties of large but *finite* systems with non-summable interactions. These methods rely essentially on the study of the mean field version of the original model. Namely, since the energy of the finite system grows faster than its size in the limit of large volumes Λ, the model is modified[(9,10,31)] by dividing the total hamiltonian by a well-chosen power of the volume $|\Lambda|$. In the case of Ising spins $\sigma_x = \pm 1$ with ferromagnetic interactions $\phi(x) = \|x\|^{-\alpha}$, $0 \leq \alpha \leq d$, this means considering the following formal identity:

$$\beta \sum_{x\neq y} \phi(x-y)\sigma_x\sigma_y = \beta|\Lambda|^{\delta} \sum_{x\neq y} \frac{\phi(x-y)}{|\Lambda|^{\delta}}\sigma_x\sigma_y. \tag{33}$$

The scaling parameter δ must be chosen as a function of α in order to obtain a well-defined thermodynamic limit for the potential $|\Lambda|^{-\delta}\phi(x-y)$, leading to a mean field inverse critical temperature $\beta_c^*(\alpha)$. Then, the "critical inverse temperature" $\beta_c(\alpha, \Lambda)$ of the real system in a finite volume Λ can be inferred to go to zero as $\beta_c(\alpha, \Lambda) \sim \beta_c^*(\alpha)|\Lambda|^{-\delta}$. It was numerically observed[(10,31)] that $\beta_c^*(\alpha)$ depends weakly on α, which has lead the authors to conjecture that all the systems with $0 \leq \alpha \leq d$ have the same thermodynamic behaviour, i.e. identical to the pure mean field case $\alpha = 0$. This "universal" mean field behaviour was then given an intelligible explanation by Vollmayr-Lee and Luijten.[32]

We now wish to present an argument in favor of our conjecture, similar in some ways to the strategy of ref.[32]. Apart from helping to understand our conjecture, it also sheds some light on the weak dependence in α observed numerically in refs.[10,31]. For simplicity we consider the case $q = 2$, i.e. the Ising model. Remember that a weaker form of Conjecture 1 is that when ϕ is non-summable, then $\beta_c(\phi_N) \to 0$ as $N \to \infty$, where $\beta_c(\phi_N)$ is the critical inverse temperature of the truncated model ϕ_N. Define

$$e_N := \sum_{x \neq 0} \phi_N(x) = \sum_{x \in B_N} \phi(x),$$

and consider the formal identity:

$$\begin{aligned} \beta \sum_{x \neq y} \phi_N(x-y)\sigma_x\sigma_y &= \beta e_N \sum_{x \neq y} \frac{\phi_N(x-y)}{e_N}\sigma_x\sigma_y \\ &\equiv \widehat{\beta}_N \sum_{x \neq y} \widehat{\phi}_N(x-y)\sigma_x\sigma_y, \end{aligned}$$

where we have defined the rescaled quantities

$$\widehat{\beta}_N := \beta e_N, \qquad \widehat{\phi}_N(x) := \frac{\phi_N(x)}{e_N}. \tag{34}$$

Since $\lim_N e_N = +\infty$ (we assume ϕ is non-summable), we have $\lim_N \widehat{\beta}_N = +\infty$. Moreover, the new potential $\widehat{\phi}_N$ has the following properties: 1) it has range at most N, 2) it is summable

$$\sum_{x \neq 0} \widehat{\phi}_N(x) = 1,$$

and 3) $\lim_N \widehat{\phi}_N(x) = 0$ for all x. Such properties remind us those of Kac potentials, which are of the form $\Phi_\gamma(x) := \gamma^d \varphi(\gamma x)$, where $\varphi(x) \geq 0$ is bounded, supported by $[-1, +1]^d$, with $\int \varphi(x)\mathrm{d}x = 1$, and $\gamma > 0$ is a small scaling parameter. Φ_γ thus has the following properties: 1) it has range γ^{-1}, 2) it is summable:

$$\int \Phi_\gamma(x)\mathrm{d}x = 1 \quad \forall \gamma > 0,$$

and 3) $\lim_{\gamma \to 0^+} \Phi_\gamma(x) = 0$ for all x. It is well known[24,26] that such potentials give, in the van der Waals limit $\gamma \to 0^+$, a justification of the van der Waals-Maxwell theory of liquid-vapor equilibrium: in this limit, the properties of the system converge to those of mean field, *regardless of the details of the function* φ. Moreover, it is knoww[8,11] that for $\gamma > 0$ the model is a good approximation to mean field: a phase transition occurs *before* reaching the mean field regime, and $\sup_{\gamma>0}\beta_c(\Phi_\gamma) < +\infty$.

It is tempting to ask whether this mean field behaviour of Kac potentials Φ_γ at small γ also holds for our potential $\widehat{\phi}_N$ at large N, and to identify the parameters γ^{-1} and N. Although $\widehat{\phi}_N$ is not obtained by a rescaling of a given function, as Φ_γ is, one can expect that for a reasonable potential ϕ (for example of the type (2)), the critical temperature $\beta_c(\widehat{\phi}_N)$ is uniformly bounded in N: $\sup_N \beta_c(\widehat{\phi}_N) < +\infty$. Therefore, if N is large enough so that $\widehat{\beta}_N > \sup_N \beta_c(\widehat{\phi}_N) \geq \beta_c(\widehat{\phi}_N)$, we have a phase transition in the truncated model ϕ_N, as predicted by our conjecture. Moreover, since we expect that the properties of the system with $\widehat{\phi}_N$ converge to those of mean field when $N \to \infty$, *independently* of the fine structure of ϕ, this again is in favor of the non-dependence on α observed in refs.[10,31]

Unfortunately, the only case where this scenario can be implemented rigorously is for the constant potential: $\phi(x) = c > 0$ for all $x \neq 0$ (or, equivalently, if $\alpha = 0$). In this case we have $\widehat{\phi}_N(\cdot) = |B_N|^{-1} 1_{\|x\|\leq N}(\cdot)$, and the correspondence $\gamma^{-1} \equiv N$ is immediate (this case is treated in detail in ref.[17]).

4.2. On the General Problem of Truncation.

We have seen that in the long-range ferromagnetic Potts model, most non-summable interactions with the symmetry (3) are pathological in the sense that in the limit $N \to \infty$,

$$\mu^{\beta,s}_{\phi_N} \Rightarrow \delta_s$$

for all $\beta > 0$ and all $s \in \{1, 2, \ldots, q\}$. A natural question is to ask whether some weak convergence also occurs in the case where ϕ *is* summable, i.e. when $\mu^{\beta,s}_{\phi}$ is well-defined. Is it that

$$\mu^{\beta,s}_{\phi_N} \Rightarrow \mu^{\beta,s}_{\phi} \tag{35}$$

in the limit $N \to \infty$? The answer to this question is clearly negative in $d = 1$. Namely, for the Ising model ($q = 2$), there exist summable potentials for which $\mu^{\beta,+}_{\phi}(\sigma_0 = +1) > \frac{1}{2}$ at low temperature,[12,18] but $\mu^{\beta,+}_{\phi_N}(\sigma_0 = +1) = 0$ for all N, as well known (see Remark 1). For $d \geq 3$, Aernout van Enter gave us the following example of a model with summable interactions in which the convergence (35) fails. Consider the Potts model with $q = 3$, $\phi(x) = e^{-\alpha\|x\|}$. It was shown recently by Biskup, Chayes and Crawford[6] that when α is small enough, this model has a first order phase transition in temperature. Namely, there exists $\beta_t > 0$ and $\delta_+ > \delta_- > 0$, such that $\mu^{\beta,1}_{\phi}(\sigma_0 = 1) \leq \delta_-$ for $\beta < \beta_t$, and $\mu^{\beta_t,1}_{\phi}(\sigma_0 = 1) \geq \delta_+$. Since truncation is equivalent to strictly raising the temperature (i.e. lowering β, see ref.[4]), we have $\mu^{\beta_t,1}_{\phi_N}(\sigma_0 = 1) \leq \delta_-$ for all N, and therefore (35) breaks down.

In the context of independent long-range percolation, the general problem of truncation is formulated as follows: assuming $P(0 \leftrightarrow \infty) > 0$, does there exist some large N such that $P_N(0 \leftrightarrow \infty) > 0$? This is believed to be true in general, with no assumption on the edge probability (other than, say, the symmetry (13), or for sparse interactions as (23)). This property is non-trivial for the following reason: the truncated measure P_N converges weakly to P, but the Portmanteau Theorem (see ref.[5]) doesn't apply, due to the fact that in the product topology, the boundary of the event $\{0 \leftrightarrow \infty\}$ has strictly positive probability (1, in fact). One can thus not conclude that $P_N(0 \leftrightarrow \infty) \to P(0 \leftrightarrow \infty)$. Our results of Section 3 and various existing results[3,16,28,30] give affirmative answer to the problem of truncation for particular cases. An interesting problem is to see if some counter-example can be found, similar to the one given above, in view of constructing a model where truncation fails.

5. PROOF OF PROPOSITION 1

The proof uses arguments which are standard in statistical mechanics. First, consider N, β, s as fixed and define, for any box Λ, $x \in \Lambda$,

$$a_\Lambda(x) := \mu^{\beta,s}_{\phi_N,\Lambda}(\sigma_x = s).$$

Due to the invariance of the interaction under translation, these numbers have the following property:

$$a_{\Lambda+y}(x+y) = a_\Lambda(x). \tag{36}$$

The main ingredient is the following lemma, which guarantees the existence of the thermodynamic limit along any increasing sequence of boxes.

Lemma 1. *Let $(\Lambda(n))_{n\geq 1}$ be any increasing sequence of boxes (not necessarily centered at the origin): $\Lambda(n) \subset \Lambda(n+1)$, $\Lambda(n) \nearrow \mathbf{Z}^d$. Then $(a_{\Lambda(n)}(x))_{n\geq 1}$ is non-increasing, i.e. $\lim_n a_{\Lambda(n)}(x)$ exists for all $x \in \mathbf{Z}^d$.*

Proof: The proof relies on the FKG inequality. Fix x and take n large enough so that $x \in \Lambda(n)$. We use the random-cluster representation of the probability $\mu^{\beta,s}_{\phi_N,\Lambda(n)}(\sigma_x = s)$. Let $E(n)$ be the set of edges of size at most N with at least one endpoint in $\Lambda(n)$, and $\partial^+\Lambda(n) = \{y \in \Lambda(n)^c : \exists x \in \Lambda(n),\ \|x-y\| \leq N\}$. Then (see refs.[1,13–15,20] for details):

$$a_{\Lambda(n)}(x) = \mu^{\beta,s}_{\phi_N,\Lambda(n)}(\sigma_x = s) = \frac{1}{q} + \frac{q-1}{q}\Phi^w_{\Lambda(n)}(x \leftrightarrow \partial^+\Lambda(n)).$$

The wired random cluster measure $\Phi^w_{\Lambda(n)}$ is defined on $\{0,1\}^{E(n)}$, with parameter q and edge probabilities $p_N(x) = 1 - \exp(-2\beta\phi_N(x))$. Let B_{n+1} denote the event in which all edges in $E(n+1)\backslash E(n)$ are open. Then

$$\begin{aligned}\Phi^{w}_{\Lambda(n)}(x \leftrightarrow \partial^{+}\Lambda(n)) &= \Phi^{w}_{\Lambda(n+1)}(x \leftrightarrow \partial^{+}\Lambda(n)|B_{n+1})\\ &= \Phi^{w}_{\Lambda(n+1)}(x \leftrightarrow \partial^{+}\Lambda(n+1)|B_{n+1})\\ &\geq \Phi^{w}_{\Lambda(n+1)}(x \leftrightarrow \partial^{+}\Lambda(n+1)).\end{aligned}$$

We used FKG inequality, since $\{x \leftrightarrow \partial^{+}\Lambda(n+1)\}$ and B_{n+1} are both increasing events. This shows that $(a_{\Lambda(n)}(x))_{n\geq 1}$ is monotone, and that the limit $n \to \infty$ exists, which proves the lemma. □

Corollary 1. *Consider the particular sequence* $\Lambda_n := [-n, +n]^d \cap \mathbf{Z}^d$. *Then* $a(x) := \lim_n a_{\Lambda_n}(x)$ *exists and is constant:* $a(x) = a(0)$ *for all* $x \in \mathbf{Z}^d$.

Proof: Fix $x \in \mathbf{Z}^d$ and consider any subsequence $(\Lambda'_n)_{n\geq 1} \subset (\Lambda_n)_{n\geq 1}$ with the property:

$$x \in \Lambda'_1 \subset \Lambda'_2 + x \subset \Lambda'_3 \subset \Lambda'_4 + x \subset \dots$$

Consider the sequence $(\Lambda^*(n))_{n\geq 1}$ defined by

$$\Lambda^*(n) := \begin{cases}\Lambda'_n & \text{if } n \text{ odd,}\\ \Lambda'_n + x & \text{if } n \text{ even.}\end{cases}$$

Then $\Lambda^*(n) \subset \Lambda^*(n+1)$ and by Lemma 1, $\lim_n a_{\Lambda^*(n)}(x)$ exists. Call this limit a. By considering the subsequences $(\Lambda^*(2k))_{k\geq 1}$ and $(\Lambda^*(2k+1))_{k\geq 1}$, we have

$$a = \lim_{k\to\infty} a_{\Lambda^*(2k)}(x) = \lim_{k\to\infty} a_{\Lambda'_{2k}+x}(x) = \lim_{k\to\infty} a_{\Lambda'_{2k}}(0) = a(0),$$

$$a = \lim_{k\to\infty} a_{\Lambda^*(2k+1)}(x) = \lim_{k\to\infty} a_{\Lambda'_{2k}}(x) = a(x),$$

where we used (36). This shows $a(x) = a(0)$. □

Proof of Proposition 1: If $\mu^{\beta,s}_{\phi_N} \Rightarrow \delta_s$ then (11) holds since $\{\sigma \in \Omega : \sigma_x = s\}$ is a cylinder containing the ground state s. Then, assume (11) holds. Let A be a cylinder. That is, there exists a finite set $D \subset \mathbf{Z}^d$ and for each $x \in D$ a set $E_x \subset \{1, 2, \dots, q\}$ such that $A = \{\sigma \in \Omega : \sigma_x \in E_x \,\forall x \in D\}$. If $A \ni s$, then

$$\mu^{\beta,s}_{\phi_N}(A^c) = \mu^{\beta,s}_{\phi_N}\big(\exists x \in D,\ \sigma_x \notin E_x\big) \leq \sum_{x\in D} \mu^{\beta,s}_{\phi_N}(\sigma_x \notin E_x) \leq \sum_{x\in D} \mu^{\beta,s}_{\phi_N}(\sigma_x \neq s). \tag{37}$$

By Corollary 1, $\mu^{\beta,s}_{\phi_N}(\sigma_x = s) = \mu^{\beta,s}_{\phi_N}(\sigma_0 = s)$. When (11) holds, $\mu^{\beta,s}_{\phi_N}(\sigma_0 = s) \to 1$ when $N \to \infty$, which gives $\mu^{\beta,s}_{\phi_N}(A^c) \to 0$. If $A \not\ni s$, the same computation (replacing A^c by A) yields $\mu^{\beta,s}_{\phi_N}(A) \to 0$. Therefore, $\mu^{\beta,s}_{\phi_N} \Rightarrow \delta_s$. □

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